

Quantization of N=1 and N=2 SUSY KdV models

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February 1, 2008

Abstract

The quantization procedure for both N=1 and N=2 supersymmetric Korteweg-de Vries (SUSY KdV) hierarchies is constructed. Namely, the quantum counterparts of the monodromy matrices, built by means of the integrated vertex operators, are shown to satisfy a specialization of reflection equation, leading to the quantum integrable theory. The relation of such models to the study of integrable perturbed superconformal and topological models is discussed.

1 Introduction

The Superconformal Field Theory (SCFT) is very important in many areas of Mathematical Physics, such as Superstring theory, where it appears as a basic part of the String Worldsheet Physics or the theory of two-dimensional solvable lattice models, e.g. tricritical Ising model as the well known example. The symmetries of SCFT are described by the generators of the Superconformal Algebra (SCA). This algebra is a supersymmetric extension of the infinite dimensional Virasoro algebra, which in its turn is a central extension of the algebra of vector fields on a circle.

There are two types of SCA which are especially important in physical problems, these are the so-called N=1 and N=2 SCA, named by the number of fermionic degrees of freedom in the string worldsheet superspace. The infinite dimensionality of this symmetry allows to look for integrable structures inside SCFT, that is infinite families of pairwise commuting integrals of motion (IM). Motivated by the “physical” reasons we impose also the condition of the supersymmetric (SUSY) invariance of these families, i.e. the SUSY generator(s) should be included in these families.

In this paper we review some results concerning the construction of these

families for both $N=1$ and $N=2$ SCFT. The key idea is that on the classical level the integrable structures are provided by the so-called second Hamiltonian structure of $N=1$ and $N=2$ SUSY KdV correspondingly. In order to build infinite family of pairwise commuting quantum IM, we need to quantize the corresponding integrable models.

Here we review the quantization procedure for both $N=1$ and $N=2$ SUSY KdV according to the results from the papers [1], [2], [3]. In the first part (Sec. 2) we consider the classical theory of SUSY $N=1$ KdV system, based on the twisted affine superalgebra $C(2)^{(2)} \simeq sl(1|2)^{(2)} \simeq osp(2|2)^{(2)}$. We introduce the supersymmetric Miura transformation and monodromy matrix associated with the corresponding L-operator. Then the auxiliary \mathbf{L} -matrices are constructed, which satisfy the quadratic Poisson bracket relation. In Sec. 4 we will show that quantum counterparts of these matrices coincide with a vertex-operator-represented quantum R-matrix. The quantum version of the Miura transformation, i.e. the free field representation of the superconformal algebra, is given in Sec. 3. In Sec. 4 the quantum $C_q(2)^{(2)}$ superalgebra [14] is introduced. Then it is shown that the corresponding quantum R-matrix can be represented by two vertex operators, satisfying Serre relations of lower Borel subalgebra of $C_q(2)^{(2)}$ and in the classical limit it coincides with \mathbf{L} -matrix. The vertex-operator-represented quantum R-matrix $\mathbf{L}^{(q)}$ satisfies the so-called RTT-relation, which gives a possibility to consider the model from a point of view of Quantum Inverse Scattering Method (QISM) [4], [5].

In the next four sections we switch to the study of the SUSY $N=2$ KdV quantization procedure. The $N=2$ KdV model is based on the $sl^{(1)}(2|1)$ affine superalgebra. The classical version of the associated monodromy matrix is represented by means of familiar P-exponential but the exponent consists not only of generators of simple roots and the squares of fermionic ones with appropriate exponential multipliers, but also generators, corresponding to more complicated composite roots (see Sec. 5). We prove that (as it was in the $N=1$ case) all these terms, corresponding to the composite roots disappear from the first iteration of the quantum generalization of the P-exponential; we also show that this “quantum” P-exponential coincide with the reduced universal R-matrix of $\hat{sl}_q(2|1)$ quantum affine superalgebra (see Sec. 6,7). We demonstrate that the supertraces of the quantum version of the monodromy matrix in different evaluation representations, the so-called transfer-matrices (by the obvious analogy with the lattice case) commute with the supersymmetry generators. Hence, these generators can be included in the family of the integrals of motion both on classical and the quantum level (see Sec. 8).

In the last section we discuss the relation of $N=1$ and $N=2$ SUSY KdV models to the superconformal/topological theories and their integrable perturbations.

2 Integrable SUSY $N=1$ KdV hierarchy

The SUSY $N=1$ KdV system can be constructed by means of the Drinfeld-Sokolov reduction applied to the $C(2)^{(2)}$ twisted affine superalgebra [6]. The

corresponding \mathcal{L} -operator has the following form:

$$\tilde{\mathcal{L}}_F = D_{u,\theta} - D_{u,\theta}\Phi(h_1 + h_2) - \lambda(e_1^+ + e_2^+ + e_1^- - e_2^-), \quad (1)$$

where $D_{u,\theta} = \partial_\theta + \theta\partial_u$ is a superderivative, the variable u lies on a cylinder of circumference 2π , θ is a Grassmann variable, $\Phi(u, \theta) = \phi(u) - \frac{i}{\sqrt{2}}\theta\xi(u)$ is a bosonic superfield; $h_1, h_2, e_1^\pm, e_2^\pm$ are the Chevalley generators of $C(2)$ with the following commutation relations:

$$\begin{aligned} [h_1, h_2] &= 0, \quad [h_1, e_2^\pm] = \pm e_2^\pm, \quad [h_2, e_1^\pm] = \pm e_1^\pm, \\ ad_{e_1^\pm}^2 e_2^\pm &= 0, \quad ad_{e_2^\pm}^2 e_1^\pm = 0, \\ [h_\alpha, e_\alpha^\pm] &= 0 \quad (\alpha = 1, 2), \quad [e_\beta^\pm, e_{\beta'}^\mp] = \delta_{\beta,\beta'} h_\beta \quad (\beta, \beta' = 1, 2), \end{aligned} \quad (2)$$

where the supercommutator $[,]$ is defined as follows: $[a, b] \equiv ad_a b \equiv ab - (-1)^{p(a)p(b)}ba$, where parity p is equal to 1 for odd elements and is equal to 0 for even ones. In the particular case of $C(2)$ $h_{1,2}$ are even and $e_{1,2}^\pm$ are odd. The operator (1) can be considered as more general one, taken in the evaluation representation of $C(2)^{(2)}$:

$$\mathcal{L}_F = D_{u,\theta} - D_{u,\theta}\Phi h_\alpha - (e_{\delta-\alpha} + e_\alpha), \quad (3)$$

where $h_\alpha, e_{\delta-\alpha}, e_\alpha$ are the Chevalley generators of $C(2)^{(2)}$ with such commutation relations:

$$\begin{aligned} [h_{\alpha_1}, h_{\alpha_0}] &= 0, \quad [h_{\alpha_0}, e_{\pm\alpha_1}] = \mp e_{\pm\alpha_1}, \quad [h_{\alpha_1}, e_{\pm\alpha_0}] = \mp e_{\pm\alpha_0}, \\ [h_{\alpha_i}, e_{\pm\alpha_i}] &= \pm e_{\pm\alpha_i}, \quad [e_{\pm\alpha_i}, e_{\mp\alpha_j}] = \delta_{i,j} h_{\alpha_i}, \quad (i, j = 0, 1), \\ ad_{e_{\pm\alpha_0}}^3 e_{\pm\alpha_1} &= 0, \quad ad_{e_{\pm\alpha_1}}^3 e_{\pm\alpha_0} = 0 \end{aligned} \quad (4)$$

where $p(h_{\alpha_{0,1}}) = 0, p(e_{\pm\alpha_{0,1}}) = 1$ and $\alpha_1 \equiv \alpha, \alpha_0 \equiv \delta - \alpha$. The Poisson brackets for the field Φ , obtained by means of the Drinfeld-Sokolov reduction are:

$$\{D_{u,\theta}\Phi(u, \theta), D_{u',\theta'}\Phi(u', \theta')\} = D_{u,\theta}(\delta(u - u')(\theta - \theta')) \quad (5)$$

and the following boundary conditions are imposed on the components of Φ : $\phi(u + 2\pi) = \phi(u) + 2\pi ip, \xi(u + 2\pi) = \pm\xi(u)$. The \mathcal{L}_F -operator is written in the Miura form, making a gauge transformation one can obtain a new superfield $\mathcal{U}(u, \theta) \equiv D_{u,\theta}\Phi(u, \theta)\partial_u\Phi(u, \theta) - D_{u,\theta}^3\Phi(u, \theta) = -\theta U(u) - i\alpha(u)/\sqrt{2}$, where U and α generate the superconformal algebra under the Poisson brackets:

$$\begin{aligned} \{U(u), U(v)\} &= \delta'''(u - v) + 2U'(u)\delta(u - v) + 4U(u)\delta'(u - v), \\ \{U(u), \alpha(v)\} &= 3\alpha(u)\delta'(u - v) + \alpha'(u)\delta(u - v), \\ \{\alpha(u), \alpha(v)\} &= 2\delta''(u - v) + 2U(u)\delta(u - v). \end{aligned} \quad (6)$$

Using one of the corresponding infinite family of IM, which are in involution under the Poisson brackets (these IM could be extracted from the monodromy

matrix of \mathcal{L}_B -operator, see below)[7]:

$$\begin{aligned} I_1^{(cl)} &= \frac{1}{2\pi} \int U(u) du, \\ I_3^{(cl)} &= \frac{1}{2\pi} \int \left(U^2(u) + \alpha(u)\alpha'(u)/2 \right) du, \\ I_5^{(cl)} &= \frac{1}{2\pi} \int \left(U^3(u) - (U')^2(u)/2 - \alpha'(u)\alpha''(u)/4 - \alpha'(u)\alpha(u)U(u) \right) du, \end{aligned} \quad (7)$$

one can obtain an evolution equation; for example, taking I_2 we get the SUSY N=1 KdV equation [8]: $\mathcal{U}_t = -\mathcal{U}_{uuu} + 3(\mathcal{U}D_{u,\theta}\mathcal{U})_u$ and in components: $U_t = -U_{uuu} - 6UU_u - \frac{3}{2}\alpha\alpha_{uu}$, $\alpha_t = -4\alpha_{uuu} - 3(U\alpha)_u$. As we have noted in the introduction one can show that the IM are invariant under supersymmetry transformation generated by $\int_0^{2\pi} du \alpha(u)$.

In order to construct the so-called monodromy matrix we introduce the \mathcal{L}_B -operator, equivalent to the \mathcal{L}_F one:

$$\mathcal{L}_B = \partial_u - \phi'(u)h_{\alpha_1} + (e_{\alpha_1} + e_{\alpha_0} - \frac{i}{\sqrt{2}}\xi h_{\alpha_1})^2 \quad (8)$$

The equivalence can be easily established if one considers the linear problem associated with the \mathcal{L}_F -operator: $\mathcal{L}_F\chi(u, \theta) = 0$ (we consider this operator acting in some representation of $C(2)^{(2)}$ and $\chi(u, \theta)$ is the vector in this representation). Then, expressing $\chi(u, \theta)$ in components: $\chi(u, \theta) = \chi_0(u) + \theta\chi_1(u)$, we find: $\mathcal{L}_B\chi_0 = 0$ and $\chi_1 = (e_{\alpha_1} + e_{\alpha_0} - \frac{i}{\sqrt{2}}\xi h_{\alpha_1})\chi_0$.

The solution to the equation $\mathcal{L}_B\chi_0 = 0$ can be written in the following way:

$$\begin{aligned} \chi_0(u) &= e^{\phi(u)h_{\alpha_1}} P \exp \int_0^u du' \left(\frac{i}{\sqrt{2}}\xi(u')e^{-\phi(u')}e_{\alpha_1} \right. \\ &\quad \left. - \frac{i}{\sqrt{2}}\xi(u')e^{\phi(u')}e_{\alpha_0} - e_{\alpha_1}^2 e^{-2\phi(u')} - e_{\alpha_0}^2 e^{2\phi(u')} - [e_{\alpha_1}, e_{\alpha_0}] \right) \eta, \end{aligned} \quad (9)$$

where η is a constant vector in the corresponding representation of $C(2)^{(2)}$. Therefore we can define the monodromy matrix in the following way:

$$\begin{aligned} \mathbf{M} &= e^{2\pi i p h_{\alpha_1}} P \exp \int_0^{2\pi} du \left(\frac{i}{\sqrt{2}}\xi(u)e^{-\phi(u)}e_{\alpha_1} \right. \\ &\quad \left. - \frac{i}{\sqrt{2}}\xi(u)e^{\phi(u)}e_{\alpha_0} - e_{\alpha_1}^2 e^{-2\phi(u)} - e_{\alpha_0}^2 e^{2\phi(u)} - [e_{\alpha_1}, e_{\alpha_0}] \right). \end{aligned} \quad (10)$$

Introducing then (as in [9]) the auxiliary \mathbf{L} -operators: $\mathbf{L} = e^{-\pi i p h_{\alpha_1}} \mathbf{M}$ we find that in the evaluation representation (when λ , the spectral parameter appears) the following Poisson bracket relation is satisfied [10]:

$$\{\mathbf{L}(\lambda) \otimes, \mathbf{L}(\mu)\} = [\mathbf{r}(\lambda\mu^{-1}), \mathbf{L}(\lambda) \otimes \mathbf{L}(\mu)], \quad (11)$$

where $\mathbf{r}(\lambda\mu^{-1})$ is trigonometric $C(2)^{(2)}$ r-matrix [11]. From this relation one obtains that the supertraces of monodromy matrices $\mathbf{t}(\lambda) = \text{str} \mathbf{M}(\lambda)$ commute

under the Poisson bracket: $\{\mathbf{t}(\lambda), \mathbf{t}(\mu)\} = 0$. Expanding $\log(\mathbf{t}(\lambda))$ in λ in the evaluation representation corresponding to the defining 3-dimensional representation of $C(2) \pi_{1/2}$ we find:

$$\log(\mathbf{t}_{1/2}(\lambda)) = - \sum_{n=1}^{\infty} c_n I_{2n-1}^{(cl)} \lambda^{-4n+2}, \quad \lambda \rightarrow \infty \quad (12)$$

where $c_1 = \frac{1}{2}$, $c_n = \frac{(2n-3)!!}{2^n n!}$ for $n > 1$. So, one can obtain the IM from the supertrace of the monodromy matrix. Using the Poisson bracket relation for these supertraces with different values of spectral parameter (see above) we find that infinite family of IM is involutive, as it was mentioned earlier.

3 Free field representation of superconformal algebra

In this section we begin to build quantum counterparts of the introduced classical objects. We will start from the quantum Miura transformation, the free field representation of the superconformal algebra [12]:

$$\begin{aligned} -\beta^2 T(u) &= : \phi'^2(u) : - (1 - \beta^2/2) \phi''(u) + \frac{1}{2} : \xi \xi'(u) : + \frac{\epsilon \beta^2}{16} \\ \frac{i^{1/2} \beta^2}{\sqrt{2}} G(u) &= \phi' \xi(u) - (1 - \beta^2/2) \xi'(u), \end{aligned} \quad (13)$$

where

$$\begin{aligned} \phi(u) &= iQ + iPu + \sum_n \frac{a_{-n}}{n} e^{inu}, \quad \xi(u) = i^{-1/2} \sum_n \xi_n e^{-inu}, \\ [Q, P] &= \frac{i}{2} \beta^2, \quad [a_n, a_m] = \frac{\beta^2}{2} n \delta_{n+m,0}, \quad \{\xi_n, \xi_m\} = \beta^2 \delta_{n+m,0}. \end{aligned} \quad (14)$$

Recall that there are two types of boundary conditions on ξ : $\xi(u+2\pi) = \pm \xi(u)$. The sign “+” corresponds to the R sector, the case when ξ is integer modded, the “-” sign corresponds to the NS sector and ξ is half-integer modded. The variable ϵ in (13) is equal to zero in the R case and equal to 1 in the NS case. One can expand $T(u)$ and $G(u)$ by modes in such a way: $T(u) = \sum_n L_{-n} e^{inu} - \frac{\hat{c}}{16}$, $G(u) = \sum_n G_{-n} e^{inu}$, where $\hat{c} = 5 - 2(\frac{\beta^2}{2} + \frac{2}{\beta^2})$ and L_n, G_m generate the superconformal algebra:

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{\hat{c}}{8}(n^3-n)\delta_{n,-m}, \quad [L_n, G_m] = (\frac{n}{2}-m)G_{m+n} \\ [G_n, G_m] &= 2L_{n+m} + \delta_{n,-m} \frac{\hat{c}}{2}(n^2-1/4). \end{aligned} \quad (15)$$

In the classical limit $c \rightarrow -\infty$ (the same is $\beta^2 \rightarrow 0$) the following substitution: $T(u) \rightarrow -\frac{\hat{c}}{4}U(u)$, $G(u) \rightarrow -\frac{\hat{c}}{2\sqrt{2}i}\alpha(u)$, $[\cdot, \cdot] \rightarrow \frac{4\pi}{i\hat{c}}\{\cdot, \cdot\}$ reduce the above algebra to

the Poisson bracket algebra of SUSY N=1 KdV theory.

Let now F_p be the highest weight module over the oscillator algebra of a_n, ξ_m with the highest weight vector (ground state) $|p\rangle$ determined by the eigenvalue of P and nilpotency condition of the action of the positive modes: $P|p\rangle = p|p\rangle$, $a_n|p\rangle = 0$, $\xi_m|p\rangle = 0$ where $n, m > 0$. In the case of the R sector the highest weight becomes doubly degenerate due to the presence of zero mode ξ_0 . So, there are two ground states $|p, +\rangle$ and $|p, -\rangle$: $|p, +\rangle = \xi_0|p, -\rangle$. Using the above free field representation of the superconformal algebra one can obtain that for generic \hat{c} and p , F_p is isomorphic to the super-Virasoro module with the highest weight vector $|p\rangle$: $L_0|p\rangle = \Delta_{NS}|p\rangle$, where $\Delta_{NS} = (p/\beta)^2 + (\hat{c} - 1)/16$ in the NS sector and module with two highest weight vectors in the Ramond case: $L_0|p, \pm\rangle = \Delta_R|p, \pm\rangle$, $\Delta_R = (p/\beta)^2 + \hat{c}/16$, $|p, +\rangle = (\beta^2/\sqrt{2}p)G_0|p, -\rangle$. The space F_p , now considered as super-Virasoro module, splits into the sum of finite-dimensional subspaces, determined by the value of L_0 : $F_p = \bigoplus_{k=0}^{\infty} F_p^{(k)}$, $L_0 F_p^{(k)} = (\Delta + k) F_p^{(k)}$. The quantum versions of local integrals of motion should act invariantly on the subspaces $F_p^{(k)}$. Thus, the diagonalization of IM reduces (in a given subspace $F_p^{(k)}$) to the finite purely algebraic problem, which however rapidly become rather complicated for large k . It should be noted also that in the case of the Ramond sector supersymmetry generator G_0 commute with IM, so IM act in $|p, +\rangle$ and $|p, -\rangle$ independently, without mixing of these two ground states (unlike the super-KdV case [13]).

4 Quantum monodromy matrix and RTT-relation

In this part of the work we will consider the quantum $C_q(2)^{(2)}$ R-matrix and show that the vertex operator representation of the lower Borel subalgebra of $C_q(2)^{(2)}$ allows to represent this R-matrix in the P-exponent like form which in the classical limit coincide with the auxiliary L-operator. $C_q(2)^{(2)}$ is a quantum superalgebra with the following commutation relations [14]:

$$\begin{aligned} [h_{\alpha_0}, h_{\alpha_1}] &= 0, \quad [h_{\alpha_0}, e_{\pm\alpha_1}] = \mp e_{\pm\alpha_1}, \quad [h_{\alpha_1}, e_{\pm\alpha_0}] = \mp e_{\pm\alpha_0}, \\ [h_{\alpha_i}, e_{\pm\alpha_i}] &= \pm e_{\pm\alpha_i} \quad (i = 0, 1), \quad [e_{\pm\alpha_i}, e_{\mp\alpha_j}] = \delta_{i,j} [h_{\alpha_i}] \quad (i, j = 0, 1), \\ [e_{\pm\alpha_1}, [e_{\pm\alpha_1}, [e_{\pm\alpha_1}, e_{\pm\alpha_0}]_q]_q]_q &= 0, \quad [[e_{\pm\alpha_1}, e_{\pm\alpha_0}]_q, e_{\pm\alpha_0}]_q = 0, \end{aligned} \quad (16)$$

where $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$, $p(h_{\alpha_{0,1}}) = 0$, $p(e_{\pm\alpha_{0,1}}) = 1$ and q-supercommutator is defined in the following way: $[e_\gamma, e_{\gamma'}]_q \equiv e_\gamma e_{\gamma'} - (-1)^{p(e_\gamma)p(e_{\gamma'})} q^{(\gamma, \gamma')} e_{\gamma'} e_\gamma$, $q = e^{i\pi \frac{\beta^2}{2}}$. The corresponding coproducts are:

$$\begin{aligned} \Delta(h_{\alpha_j}) &= h_{\alpha_j} \otimes 1 + 1 \otimes h_{\alpha_j}, \quad \Delta(e_{\alpha_j}) = e_{\alpha_j} \otimes q^{h_{\alpha_j}} + 1 \otimes e_{\alpha_j}, \\ \Delta(e_{-\alpha_j}) &= e_{-\alpha_j} \otimes 1 + q^{-h_{\alpha_j}} \otimes e_{-\alpha_j}. \end{aligned} \quad (17)$$

The associated R-matrix can be expressed in such a way [14]: $R = KR_+R_0R_-$, where $K = q^{h_\alpha \otimes h_\alpha}$, $R_+ = \prod_{n \geq 0} R_{n\delta + \alpha}$, $R_- = \prod_{n \geq 1} R_{n\delta - \alpha}$, $R_0 = \exp((q - q^{-1}) \sum_{n > 0} d(n) e_{n\delta} \otimes e_{-n\delta})$. Here $R_\gamma = \exp_{(-q^{-1})}(A(\gamma)(q - q^{-1})(e_\gamma \otimes e_{-\gamma}))$ and

coefficients A and d are defined as follows: $A(\gamma) = \{(-1)^n \text{ if } \gamma = n\delta + \alpha; (-1)^{n-1} \text{ if } \gamma = n\delta - \alpha\}$, $d(n) = \frac{n(q-q^{-1})}{q^n - q^{-n}}$. The generators $e_{n\delta}$, $e_{n\delta \pm \alpha}$ are defined via the q -commutators of Chevalley generators. In the following we will need the expressions only for simplest ones: $e_\delta = [e_{\alpha_0}, e_{\alpha_1}]_{q^{-1}}$ and $e_{-\delta} = [e_{-\alpha_1}, e_{-\alpha_0}]_q$. The elements $e_{n\delta \pm \alpha}$ are expressed as multiple commutators of e_δ with corresponding Chevalley generators, $e_{n\delta}$ ones have more complicated form [14].

Let's introduce the reduced R-matrix $\bar{R} \equiv K^{-1}R$. Using all previous information one can write \bar{R} as $\bar{R}(\bar{e}_{\alpha_i}, \bar{e}_{-\alpha_i})$, where $\bar{e}_{\alpha_i} = e_{\alpha_i} \otimes 1$ and $\bar{e}_{-\alpha_i} = 1 \otimes e_{-\alpha_i}$, because it is represented as power series of these elements. After this necessary background we will introduce vertex operators and using the fact that they satisfy the Serre relations of the lower Borel subalgebra of $C_q(2)^{(2)}$ we will prove that the reduced R-matrix, represented by the vertex operators has the properties of the P-exponent. So, the vertex operators are:

$$V_1 = \frac{1}{q^{-1} - q} \int d\theta \int_{u_2}^{u_1} du : e^{-\Phi} :, \quad V_0 = \frac{1}{q^{-1} - q} \int d\theta \int_{u_2}^{u_1} du : e^{\Phi} :, \quad (18)$$

where $2\pi \geq u_1 \geq u_2 \geq 0$, $\Phi = \phi(u) - \frac{i}{\sqrt{2}}\theta\xi(u)$ is a superfield and normal ordering here means that $: e^{\pm\phi(u)} := \exp\left(\pm \sum_{n=1}^{\infty} \frac{a_{-n}}{n} e^{inu}\right) \exp\left(\pm i(Q + Pu)\right) \exp\left(\mp \sum_{n=1}^{\infty} \frac{a_n}{n} e^{-inu}\right)$. One can show via the standard contour technique that these operators satisfy the same commutation relations as e_{α_1} , e_{α_0} correspondingly.

Then, following [15] one can show (using the fundamental property of the universal R-matrix: $(I \otimes \Delta)R = R^{13}R^{12}$) that the reduced R-matrix has the following property:

$$\bar{R}(\bar{e}_{\alpha_i}, e'_{-\alpha_i} + e''_{-\alpha_i}) = \bar{R}(\bar{e}_{\alpha_i}, e'_{-\alpha_i})\bar{R}(\bar{e}_{\alpha_i}, e''_{-\alpha_i}), \quad (19)$$

where $e'_{-\alpha_i} = 1 \otimes q^{-h_{\alpha_i}} \otimes e_{-\alpha_i}$, $e''_{-\alpha_i} = 1 \otimes e_{-\alpha_i} \otimes 1$, $\bar{e}_{\alpha_i} = e_{\alpha_i} \otimes 1 \otimes 1$. The commutation relations between them are:

$$\begin{aligned} e'_{-\alpha_i} \bar{e}_{\alpha_j} &= -\bar{e}_{\alpha_j} e'_{-\alpha_i}, & e''_{-\alpha_i} \bar{e}_{\alpha_j} &= -\bar{e}_{\alpha_j} e''_{-\alpha_i}, \\ e'_{-\alpha_i} e''_{-\alpha_j} &= -q^{b_{ij}} e''_{-\alpha_j} e'_{-\alpha_i}, \end{aligned} \quad (20)$$

where b_{ij} is the symmetric matrix with the following elements: $b_{00} = b_{11} = -b_{01} = 1$. Now, denoting $\bar{\mathbf{L}}^{(q)}(u_2, u_1)$ the reduced R-matrix with $e_{-\alpha_i}$ represented by V_i , we find, using the above property of \bar{R} with $e'_{-\alpha_i}$ replaced by appropriate vertex operators: $\bar{\mathbf{L}}^{(q)}(u_3, u_1) = \bar{\mathbf{L}}^{(q)}(u_2, u_1)\bar{\mathbf{L}}^{(q)}(u_3, u_2)$ with $u_1 \geq u_2 \geq u_3$. So, $\bar{\mathbf{L}}^{(q)}$ has the property of P-exponent. But because of singularities in the operator products of vertex operators it can not be written in the usual P-ordered form. Thus, we propose a new notion, the quantum P-exponent:

$$\bar{\mathbf{L}}^{(q)}(u_1, u_2) = P \exp^{(q)} \int_{u_2}^{u_1} du \int d\theta (e_{\alpha_1} : e^{-\Phi} : + e_{\alpha_0} : e^{\Phi} :). \quad (21)$$

Introducing new object: $\mathbf{L}^{(q)} \equiv e^{i\pi P h_{\alpha_1}} \bar{\mathbf{L}}^{(q)}(0, 2\pi)$, which coincides with R-matrix with $1 \otimes h_{\alpha_1}$ replaced by $2P/\beta^2$ and $1 \otimes e_{-\alpha_1}$, $1 \otimes e_{-\alpha_0}$ replaced by V_1 and V_0 (with integration from 0 to 2π) correspondingly, we find that it satisfies the well-known RTT-relation:

$$\begin{aligned} \mathbf{R}(\lambda\mu^{-1}) \left(\mathbf{L}^{(q)}(\lambda) \otimes \mathbf{I} \right) \left(\mathbf{I} \otimes \mathbf{L}^{(q)}(\mu) \right) \\ = \left(\mathbf{I} \otimes \mathbf{L}^{(q)}(\mu) \right) \left(\mathbf{L}^{(q)}(\lambda) \otimes \mathbf{I} \right) \mathbf{R}(\lambda\mu^{-1}), \end{aligned} \quad (22)$$

where the dependence on λ, μ means that we are considering $\mathbf{L}^{(q)}$ -operators in the evaluation representation of $C_q(2)^{(2)}$. The quantum counterpart of the monodromy matrix $\mathbf{M}^{(q)} = e^{\sum_{i=1,2} \pi i p_i h_{\alpha_i}} \bar{\mathbf{L}}^{(q)}$ satisfies the reflection equation [21]:

$$\tilde{\mathbf{R}}_{12}(\lambda\mu^{-1}) \mathbf{M}_1^{(q)}(\lambda) F_{12}^{-1} \mathbf{M}_2^{(q)}(\mu) = \mathbf{M}_2^{(q)}(\mu) F_{12}^{-1} \mathbf{M}_1^{(q)}(\lambda) \mathbf{R}_{12}(\lambda\mu^{-1}), \quad (23)$$

where $F = K^{-1}$, the Cartan's factor from the universal R-matrix, $\tilde{\mathbf{R}}_{12}(\lambda\mu^{-1}) = F_{12}^{-1} \mathbf{R}_{12}(\lambda\mu^{-1}) F_{12}$ and labels 1,2 denote the position of multipliers in the tensor product. Thus the supertraces of monodromy matrices, the transfer matrices $\mathbf{t}^{(q)}(\lambda) \equiv \text{str}(\mathbf{M}^{(q)}(\lambda))$ commute

$$[\mathbf{t}^{(q)}(\lambda), \mathbf{t}^{(q)}(\mu)] = 0, \quad (24)$$

giving the quantum integrability. Now we will show that in the classical limit ($q \rightarrow 1$) the $\mathbf{L}^{(q)}$ -operator will give the auxiliary \mathbf{L} -matrix defined in the Sec. 2. We will use the P-exponent property of $\bar{\mathbf{L}}^{(q)}(0, 2\pi)$. Let's decompose $\bar{\mathbf{L}}^{(q)}(0, 2\pi)$ in the following way: $\bar{\mathbf{L}}^{(q)}(0, 2\pi) = \lim_{N \rightarrow \infty} \prod_{m=1}^N \bar{\mathbf{L}}^{(q)}(x_{m-1}, x_m)$, where we divided the interval $[0, 2\pi]$ into infinitesimal intervals $[x_m, x_{m+1}]$ with $x_{m+1} - x_m = \epsilon = 2\pi/N$. Let's find the terms that can give contribution of the first order in ϵ in $\bar{\mathbf{L}}^{(q)}(x_{m-1}, x_m)$. In this analysis we will need the operator product expansion of vertex operators:

$$\begin{aligned} \xi(u)\xi(u') &= -\frac{i\beta^2}{(iu - iu')} + \sum_{k=1}^{\infty} c_k(u)(iu - iu')^k, \\ :e^{a\phi(u)}::e^{b\phi(u')} &:= (iu - iu')^{\frac{ab\beta^2}{2}} (:e^{(a+b)\phi(u)}: + \sum_{k=1}^{\infty} d_k(u)(iu - iu')^k), \end{aligned} \quad (25)$$

where $c_k(u)$ and $d_k(u)$ are operator-valued functions of u . Now one can see that only two types of terms can give the contribution of the order ϵ in $\bar{\mathbf{L}}^{(q)}(x_{m-1}, x_m)$ when $q \rightarrow 1$. The first type consists of operators of the first order in V_i and the second type is formed by the operators, quadratic in V_i , which give contribution of the order $\epsilon^{1 \pm \beta^2}$ by virtue of operator product expansion. Let's look on the terms of the second type in detail. At first we consider the terms appearing from the R_0 -part of R-matrix, represented by vertex operators:

$$\frac{e_\delta}{2(q - q^{-1})} \left(\int_{x_{m-1}}^{x_m} du_1 : e^{-\phi} : \xi(u_1 - i0) \int_{x_{m-1}}^{x_m} du_2 : e^\phi : \xi(u_2 + i0) + \right.$$

$$q^{-1} \int_{x_{m-1}}^{x_m} du_2 : e^\phi : \xi(u_2 - i0) \int_{x_{m-1}}^{x_m} du_1 : e^{-\phi} : \xi(u_1 + i0) \Big) \quad (26)$$

Neglecting the terms, which give rise to $O(\epsilon^2)$ contribution, we obtain, using the operator products of vertex operators:

$$\begin{aligned} & \frac{e_\delta}{2(q - q^{-1})} \left(\int_{x_{m-1}}^{x_m} du_1 \int_{x_{m-1}}^{x_m} du_2 \frac{-i\beta^2}{(i(u_1 - u_2 - i0))^{\frac{\beta^2}{2}+1}} + \right. \\ & \left. q^{-1} \int_{x_{m-1}}^{x_m} du_2 \int_{x_{m-1}}^{x_m} du_1 \frac{-i\beta^2}{(i(u_2 - u_1 - i0))^{\frac{\beta^2}{2}+1}} \right) \end{aligned} \quad (27)$$

In the $\beta^2 \rightarrow 0$ limit we get: $\frac{[e_{\alpha_1}, e_{\alpha_0}]}{2i\pi} \int_{x_{m-1}}^{x_m} du_1 \int_{x_{m-1}}^{x_m} du_2 \left(\frac{1}{u_1 - u_2 + i0} - \frac{1}{u_1 - u_2 - i0} \right)$ which, by the well known formula: $\frac{1}{x+i0} = P\frac{1}{x} - i\pi\delta(x)$ gives: $-\int_{x_{m-1}}^{x_m} du [e_{\alpha_1}, e_{\alpha_0}]$. Another terms arise from the R_+ and R_- parts of R-matrix and are very similar to each other:

$$\begin{aligned} & \frac{e_{\alpha_0}^2}{2(2)(-q^{-1})} \int_{x_{m-1}}^{x_m} du_1 : e^\phi : \xi(u_1 - i0) \int_{x_{m-1}}^{x_m} du_2 : e^\phi : \xi(u_2 + i0), \\ & \frac{e_{\alpha_1}^2}{2(2)(-q^{-1})} \int_{x_{m-1}}^{x_m} du_1 : e^{-\phi} : \xi(u_1 - i0) \int_{x_{m-1}}^{x_m} du_2 : e^{-\phi} : \xi(u_2 + i0). \end{aligned} \quad (28)$$

The integrals can be reduced to the ordered ones:

$$\begin{aligned} & \frac{e_{\alpha_0}^2}{2} \int_{x_{m-1}}^{x_m} du_1 : e^\phi : \xi(u_1) \int_{x_{m-1}}^{u_1} du_2 : e^\phi : \xi(u_2), \\ & \frac{e_{\alpha_1}^2}{2} \int_{x_{m-1}}^{x_m} du_1 : e^{-\phi} : \xi(u_1) \int_{x_{m-1}}^{u_1} du_2 : e^{-\phi} : \xi(u_2). \end{aligned} \quad (29)$$

Following [13] we find that their contribution (of order ϵ) in the classical limit is:

$$-e_{\alpha_0}^2 \int_{x_{m-1}}^{x_m} du e^{2\phi(u)}, \quad -e_{\alpha_1}^2 \int_{x_{m-1}}^{x_m} du e^{-2\phi(u)}. \quad (30)$$

Gathering now all the terms of order ϵ we find:

$$\begin{aligned} \bar{\mathbf{L}}^{(1)}(x_{m-1}, x_m) &= 1 + \int_{x_{m-1}}^{x_m} du \left(\frac{i}{\sqrt{2}} \xi(u) e^{-\phi(u)} e_{\alpha_1} - \right. \\ & \left. \frac{i}{\sqrt{2}} \xi(u) e^{\phi(u)} e_{\alpha_0} - e_{\alpha_1}^2 e^{-2\phi(u)} - e_{\alpha_0}^2 e^{2\phi(u)} - [e_{\alpha_1}, e_{\alpha_0}] \right) + O(\epsilon^2) \end{aligned} \quad (31)$$

and collecting all $\bar{\mathbf{L}}^{(1)}(x_{m-1}, x_m)$ we find that $\bar{\mathbf{L}}^{(1)} = e^{-i\pi p h_{\alpha_1}} \mathbf{L}$. Therefore $\mathbf{L}^{(1)} = \mathbf{L}$.

5 $N = 2$ SUSY KdV hierarchy in nonstandard form

The matrix L-operator of the N=2 SUSY KdV has the following explicit form [16]:

$$\mathcal{L}_F = D - \begin{pmatrix} D\Phi_1 & 1 & 0 \\ \lambda & D\Phi_1 - D\Phi_2 & 1 \\ \lambda D(\Phi_1 - \Phi_2) & \lambda & -D\Phi_2 \end{pmatrix},$$

where $D = \partial_\theta + \theta\partial_u$ and u is a variable on a cylinder of circumference 2π and $\Phi_i(u, \theta) = \phi_i(u) - \frac{i}{\sqrt{2}}\theta\xi_i(u)$ are the superfields with the following Poisson brackets:

$$\begin{aligned} \{\Phi_i(u_1, \theta_1), \Phi_i(u_2, \theta_2)\} &= 0 \quad (i = 1, 2) \\ \{D_{u_1, \theta_1}\Phi_1(u_1, \theta_1), D_{u_2, \theta_2}\Phi_2(u_2, \theta_2)\} &= -D_{u_1, \theta_1}(\delta(u_1 - u_2)(\theta_1 - \theta_2)) \end{aligned} \quad (32)$$

This is odd L-operator, related to the $sl^{(1)}(2|1)$ affine superalgebra (the elements in the associated column of the representation vector are graded from upper to the lower in such an order: even, odd, even). There is also one (canonical) related to the $sl^{(1)}(2|2)$ [6]. The quantization of this higher rank case can be performed by means of the procedure outlined in [2] and does not add something new in the quantization procedure with respect to [13], [2]. Moreover if we look forward to study and possibly solve the corresponding integrable quantum model we will find that the associated representation theory of $sl^{(1)}(2|2)$ (both classical and quantum) is very complicated. However the introduced lower rank form allows us to work with much more simple representation theory of $sl^{(1)}(2|1)$ and, as we will see, it provides very interesting features of the quantization formalism.

In order to build the scalar L-operator related with this matrix one, let's consider the corresponding linear problem ($\mathcal{L}\Psi = 0$) and express the second and third element in the vector Ψ in terms of the first (upper) one. The linear equation for this element is the following one:

$$((D + D\Phi_1)(D - D(\Phi_1 - \Phi_2))(D - D\Phi_2) + 2\lambda D)\Psi_1 = 0 \quad (33)$$

Therefore the scalar linear operator is:

$$L = D^3 + (\mathcal{V} + 2\lambda)D + \mathcal{U}, \quad (34)$$

where the Miura map is given by:

$$\begin{aligned} \mathcal{V} &= -D\Phi_2 D\Phi_1 + \partial\Phi_2 - \partial\Phi_1 = \frac{i}{\sqrt{2}}\theta(\alpha^+ - \alpha^-) + V, \\ \mathcal{U} &= -\partial\Phi_2 D\Phi_1 - D\partial\Phi_1 = \theta U + \frac{i}{\sqrt{2}}\alpha^+ \\ U &= -\phi'_1\phi'_2 - \frac{1}{2}\xi_1\xi'_2 - \phi''_1, \quad \alpha^+ = \xi_1\phi'_2 + \xi'_1, \\ \alpha^- &= \xi_2\phi'_1 + \xi'_2, \quad V = \phi'_2 - \phi'_1 + \frac{1}{2}\xi_2\xi_1 \end{aligned} \quad (35)$$

and the introduced objects U, V, α^\pm satisfy the N=2 superconformal algebra under the Poisson brackets:

$$\begin{aligned}
\{U(u), \alpha^+(v)\} &= -\alpha'^+(u)\delta(u-v) - 2\alpha^+(u)\delta'(u-v), \\
\{U(u), \alpha^-(v)\} &= -\alpha^-(u)\delta'(u-v), \\
\{\alpha^+(u), \alpha^-(v)\} &= -2U(u)\delta(u-v) - 2V(u)\delta'(u-v) - 2\delta''(u-v), \\
\{V(u), \alpha^+(v)\} &= -\alpha^+(u)\delta(u-v), \\
\{V(u), \alpha^-(v)\} &= \alpha^-(u)\delta(u-v), \\
\{U(u), V(v)\} &= -V(u)\delta'(u-v), \\
\{V(u), V(v)\} &= -2\delta'(u-v), \\
\{U(u), U(v)\} &= -U'(u)\delta(u-v) - 2U(u)\delta'(u-v).
\end{aligned} \tag{36}$$

Now let's rewrite the above L-operator in the purely algebraic form. That is:

$$\begin{aligned}
\mathcal{L}_F &= D - (h_{\alpha_1} D\Phi_1 + h_{\alpha_2} D\Phi_2 + e_{\alpha_1} + e_{\alpha_2} + [e_{\alpha_2}, e_{\alpha_0}] + \\
&\quad [e_{\alpha_0}, e_{\alpha_1}] + D(\Phi_1 - \Phi_2)e_{\alpha_0})
\end{aligned} \tag{37}$$

where $h_{\alpha_i}, e_{\alpha_i}$ are the generators of the upper Borel algebra of $sl^{(1)}(2|1)$, with the following commutation relations:

$$\begin{aligned}
[e_{\alpha_i}, e_{-\alpha_j}] &= \delta_{i,j} h_{\alpha_i} \quad (i = 0, 1, 2), \quad [h_{\alpha_j}, e_{\pm\alpha_i}] = \pm e_{\pm\alpha_i} \quad (i = 1, 2, i \neq j) \\
[h_{\alpha_j}, e_{\pm\alpha_0}] &= \mp e_{\pm\alpha_0} \quad (i = 1, 2), \quad [h_{\alpha_0}, e_{\pm\alpha_i}] = \mp e_{\pm\alpha_i} \quad (i = 1, 2) \\
[h_{\alpha_i}, e_{\pm\alpha_i}] &= 0 \quad (i = 1, 2), \quad [h_{\alpha_0}, e_{\pm\alpha_0}] = \pm 2e_{\pm\alpha_0}, \\
ad_{e_{\pm\alpha_i}}^2 e_{\pm\alpha_j} &= 0, \quad [e_{\pm\alpha_k}, e_{\pm\alpha_k}] = 0 \quad (k = 1, 2)
\end{aligned} \tag{38}$$

where $e_{\pm\alpha_i}$ are odd generators for $i = 1, 2$ and even for $i = 0$; $[\cdot, \cdot]$ means the supercommutator.

The symmetrized Cartan's matrix $b_{ij} = (\alpha_i, \alpha_j)$ corresponding to the given affine superalgebra is equal to:

$$b = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \tag{39}$$

Throughout the paper we will use only the evaluation representations (this means that central charge of the corresponding affine algebra in these representations is equal to zero). In order to write a monodromy matrix we need to consider the equivalent bosonic L-operator. That is, expressing the linear problem associated with the operator (34) in the following way: $\mathcal{L}\Psi = (\partial_\theta + \theta\partial_u + N_1 + \theta N_0)(\Psi_0 + \theta\Psi_1)$, it is easy to rewrite it as a bosonic linear problem on Ψ_0 : $\mathcal{L}_B\Psi_0 \equiv (\partial_u + N_1^2 + N_0)\Psi_0 = 0$ and $\Psi_1 = -N_1\Psi_0$. where $N_1 = \frac{i}{\sqrt{2}}\xi_1 h_{\alpha_1} + \frac{i}{\sqrt{2}}\xi_2 h_{\alpha_2} - e_{\alpha_1} - e_{\alpha_2} - [e_{\alpha_2}, e_{\alpha_0}] - [e_{\alpha_0}, e_{\alpha_1}] + \frac{i}{\sqrt{2}}(\xi_1 - \xi_2)e_{\alpha_0}$ and $N_0 = -\phi'_1 h_{\alpha_1} - \phi'_2 h_{\alpha_2} - (\phi'_1 - \phi'_2)e_{\alpha_0}$. that is:

$$\begin{aligned}
\mathcal{L}_B &= \partial_u + \left(\frac{i}{\sqrt{2}}\xi_1 h_{\alpha_1} + \frac{i}{\sqrt{2}}\xi_2 h_{\alpha_2} - [e_{\alpha_2}, e_{\alpha_0}] - [e_{\alpha_0}, e_{\alpha_1}]\right. \\
&\quad \left.- e_{\alpha_1} - e_{\alpha_2} + \frac{i}{\sqrt{2}}(\xi_1 - \xi_2)e_{\alpha_0}\right)^2 - (\phi'_1 - \phi'_2)e_{\alpha_0} - \phi'_1 h_{\alpha_1} - \phi'_2 h_{\alpha_2}
\end{aligned} \tag{40}$$

Considering the associated linear problem one can write the solution in the following way:

$$\chi(u) = e^{\sum_{i=1,2} \phi_i(u) h_{\alpha_i}} \text{Pexp} \int_0^u du' \left(\sum_{k=0,1,2} W_{\alpha_k}(u') e_{\alpha_k} + K(u') \right) \quad (41)$$

where $W_{\alpha_i} = \int d\theta e^{-\Phi_i}$ for $i = 1, 2$ and $W_{\alpha_0} = \int d\theta (D\Phi_1 - D\Phi_2) e^{\Phi_1 + \Phi_2}$,

$$K(u) = -\frac{i}{\sqrt{2}} \xi_2 [e_{\alpha_2}, e_{\alpha_0}] e^{\phi_2} - \frac{i}{\sqrt{2}} \xi_1 [e_{\alpha_0}, e_{\alpha_1}] e^{\phi_1} - [e_{\alpha_1}, e_{\alpha_2}] e^{-\phi_1 - \phi_2} - \\ [[e_{\alpha_0}, e_{\alpha_1}], [e_{\alpha_2}, e_{\alpha_0}]] e^{\phi_1 + \phi_2} - [e_{\alpha_2}, [e_{\alpha_0}, e_{\alpha_1}]] - [e_{\alpha_1}, [e_{\alpha_2}, e_{\alpha_0}]]. \quad (42)$$

We can define then the monodromy matrix on the interval $[0, 2\pi]$ with quasiperiodic boundary conditions ($\phi_i(u + 2\pi) = \phi_i(u) + 2\pi i p_i$, $\xi_i(u + 2\pi) = \pm \xi_i(u)$) in the following way:

$$\mathbf{M} = e^{\sum_{i=1,2} 2i\pi p_i h_{\alpha_i}} \text{Pexp} \int_0^{2\pi} du \left(\sum_{k=0,1,2} W_{\alpha_k}(u) e_{\alpha_k} + K(u) \right) \quad (43)$$

The reason of separating the terms in the K -part and the covariant exponential superfield part is that in the quantum case (see below) the complicated and noncovariant K -part disappears from the expression of the quantum monodromy matrix. As in the case of the standard KdV models let's define the auxiliary L-operators

$$\mathbf{L} = e^{-\sum_{i=1,2} i\pi p_i h_{\alpha_i}} \mathbf{M}, \quad (44)$$

which satisfy the quadratic r-matrix relation [10]:

$$\{\mathbf{L}(\lambda), \otimes \mathbf{L}(\mu)\} = [\mathbf{r}(\lambda\mu^{-1}), \mathbf{L}(\lambda) \otimes \mathbf{L}(\mu)] \quad (45)$$

where we have restored the dependence on the spectral parameters λ, μ corresponding to some evaluation representations. Here $\mathbf{r}(\lambda\mu^{-1})$ is a classical trigonometric r-matrix associated with $sl^{(1)}(2|1)$ [11]. As usual this leads to the classical integrability relation:

$$\{\mathbf{t}(\lambda), \mathbf{t}(\mu)\} = 0 \quad (46)$$

where $\mathbf{t}(\lambda) = \text{str} \mathbf{M}(\lambda)$ and the supertrace is taken in some evaluation representation of $sl^{(1)}(2|1)$.

6 Quantum R-matrix and the Cartan-Weyl basis for $sl_q^{(1)}(2|1)$

The quantum algebra $sl_q^{(1)}(2|1)$ has the following commutation relations:

$$\begin{aligned} [e_{\alpha_i}, e_{-\alpha_j}] &= \delta_{i,j} [h_{\alpha_i}] \quad (i = 0, 1, 2), \quad [h_{\alpha_j}, e_{\pm\alpha_i}] = \pm e_{\pm\alpha_i} \quad (i = 1, 2, i \neq j) \\ [h_{\alpha_j}, e_{\pm\alpha_0}] &= \mp e_{\pm\alpha_0} \quad (i = 1, 2), \quad [h_{\alpha_0}, e_{\pm\alpha_i}] = \mp e_{\pm\alpha_i} \quad (i = 1, 2) \\ [h_{\alpha_i}, e_{\pm\alpha_i}] &= 0 \quad (i = 1, 2), \quad [h_{\alpha_0}, e_{\pm\alpha_0}] = \pm 2e_{\pm\alpha_0}, \\ [e_{\pm\alpha_i}, [e_{\pm\alpha_i}, e_{\pm\alpha_j}]_q]_q &= 0, \quad e_{\alpha_k}^2 = 0 \quad (k = 1, 2). \end{aligned} \quad (47)$$

where the quantum supercommutator is $[e_\alpha, e_\beta]_q = e_\alpha e_\beta - (-1)^{p(\alpha)p(\beta)} q^{(\alpha, \beta)} e_\beta e_\alpha$ and $[x] = \frac{q^x - q^{-x}}{q - q^{-1}}$.

The expression for the quantum R-matrix for quantum affine superalgebras is [18]:

$$R = K \bar{R} = K \left(\prod_{\alpha \in \Delta_+}^{\rightarrow} R_\alpha \right), \quad (48)$$

where \bar{R} is a reduced R-matrix and R_α are defined by the formulae:

$$R_\alpha = \exp_{q_\alpha^{-1}}((-1)^{p(\alpha)}(q - q^{-1})(a(\alpha))^{-1}(e_\alpha \otimes e_{-\alpha})) \quad (49)$$

for real roots and

$$R_{n\delta} = \exp((q - q^{-1}) \left(\sum_{i,j}^{mult} c_{ij}(n) e_{n\delta}^{(i)} \otimes e_{-n\delta}^{(j)} \right)) \quad (50)$$

for pure imaginary roots. Here Δ_+ is the reduced positive root system (the bosonic roots which are two times fermionic roots are excluded) and q-exponential is defined as usual:

$$\exp_q(x) = \sum_{n=0}^{\infty} x^n / (n)_q!,$$

where

$$(n)_q = (q^n - 1) / (q - 1).$$

The generators corresponding to the composite roots and the ordering in (48) are defined according to the construction of the Cartan-Weyl basis (see e.g. [18]). The Cartan factor K in the case of the $sl_q^{(1)}(1|2)$ is equal to :

$$K = q^{h_{\alpha_1} \otimes h_{\alpha_2} + h_{\alpha_2} \otimes h_{\alpha_1}}$$

The $a(\alpha), c_{ij}(n), d_{ij}(n)$ coefficients are defined as follows:

$$[e_\gamma, e_{-\gamma}] = a(\gamma)(k_\gamma - k_\gamma^{-1})(q - q^{-1}),$$

$$[e_{n\delta}^{(i)}, e_{-n\delta}^{(j)}] = d_{ij}(n)(q^{nh_\delta} - q^{-nh_\delta}) / (q - q^{-1})$$

and $c_{ij}(n)$ is an inverse matrix to $d_{ij}(n)$. The first few generators corresponding to composite roots and constructed by means of the mentioned procedure are:

$$\begin{aligned} e_{\alpha_1 + \alpha_2} &= [e_{\alpha_1}, e_{\alpha_2}]_{q^{-1}} \\ e_{\delta - \alpha_1} &\equiv e_{\alpha_0 + \alpha_2} = [e_{\alpha_0}, e_{\alpha_2}]_{q^{-1}}, \quad e_{\delta - \alpha_2} \equiv e_{\alpha_0 + \alpha_1} = [e_{\alpha_1}, e_{\alpha_0}]_{q^{-1}} \\ e_\delta^{(1)} &= [[e_{\alpha_0}, e_{\alpha_2}]_{q^{-1}}, e_{\alpha_1}], \quad e_\delta^{(2)} = [[e_{\alpha_1}, e_{\alpha_0}]_{q^{-1}}, e_{\alpha_2}] \\ e_{2\delta - \alpha_1 - \alpha_2} &= [e_{\delta - \alpha_2}, e_{\delta - \alpha_1}]_{q^{-1}} \\ e_{-\alpha_1 - \alpha_2} &= [e_{-\alpha_2}, e_{-\alpha_1}]_q \\ e_{-\delta + \alpha_1} &\equiv e_{-\alpha_0 - \alpha_2} = [e_{-\alpha_2}, e_{-\alpha_0}]_q, \quad e_{-\delta + \alpha_2} \equiv e_{-\alpha_0 - \alpha_1} = [e_{-\alpha_0}, e_{-\alpha_1}]_q \\ e_{-\delta}^{(1)} &= [[e_{-\alpha_2}, e_{-\alpha_0}]_q, e_{-\alpha_1}], \quad e_{-\delta}^{(2)} = [[e_{-\alpha_0}, e_{-\alpha_1}]_q, e_{-\alpha_2}] \\ e_{-2\delta + \alpha_1 + \alpha_2} &= [e_{-\delta + \alpha_1}, e_{-\delta + \alpha_2}]_q \end{aligned} \quad (51)$$

7 The Construction of the Quantum Monodromy Matrix

In this section the quantum version of the monodromy matrix introduced in Sec. 2 will be constructed. It will be shown that in classical limit one obtains the right answer, that is the expression (43).

At first let's consider the quantum versions of the integrated exponentials of the fields (vertex operators):

$$\begin{aligned} W_{\alpha_i} &= \int d\theta : e^{-\Phi_i} : \equiv \frac{i}{\sqrt{2}} \xi_i : e^{-\phi_i} : \quad (i = 1, 2), \\ W_{\alpha_0} &= \int d\theta : (D\Phi_1 - D\Phi_2) e^{\Phi_1 + \Phi_2} : \equiv e^{\phi_1 + \phi_2} (\phi'_1 - \phi'_2 + \xi_1 \xi_2) : . \end{aligned} \quad (52)$$

One can express $\Phi_1 = \frac{i\Phi_+ + \Phi_-}{\sqrt{2}}$, $\Phi_2 = \frac{i\Phi_+ - \Phi_-}{\sqrt{2}}$, where as usual:

$$\begin{aligned} \phi_{\pm}(u) &= iQ^{\pm} + iP^{\pm}u + \sum_n \frac{a_{\pm n}^{\pm}}{n} e^{inu}, \quad \xi_{\pm}(u) = i^{-1/2} \sum_n \xi_n^{\pm} e^{-inu}, \\ [Q^{\pm}, P^{\pm}] &= \frac{i}{2} \beta^2, \quad [a_n^{\pm}, a_m^{\pm}] = \frac{\beta^2}{2} n \delta_{n+m, 0}, \quad \{\xi_n^{\pm}, \xi_m^{\pm}\} = \beta^2 \delta_{n+m, 0}. \end{aligned}$$

and the normal ordering in (52) is defined as usual:

$$: e^{c\phi_{\pm}(u)} : = \exp \left(c \sum_{n=1}^{\infty} \frac{a_{\pm n}^{\pm}}{n} e^{inu} \right) \exp \left(ci(Q^{\pm} + P^{\pm}u) \right) \exp \left(-c \sum_{n=1}^{\infty} \frac{a_{\pm n}^{\pm}}{n} e^{-inu} \right).$$

Here the a_n^{\pm} operators with n negative are placed to the left and with n positive to the right. The vertex operators (52) integrated from u_1 to u_2 satisfy the quantum Serre and “non Serre” relations (47) for the lower Borel subalgebra with $q = e^{\frac{i\pi\beta^2}{2}}$ [19].

The proof of this proposition is nontrivial because the usual proof of the Serre relations given in the papers [19] for the bosonic case, based on the transformation of the product of the integrated vertex operators to the ordered integrals is not valid here. This happens because of singularities generated by the fermion fields in the corresponding operator products. But there is another way of proof, which uses usual in CFT technique of contour integration and analytic continuation of the operator products of nonlocal vertex operators [20]. This proof is appropriate also for the case of quantum affine superalgebra and corresponding vertex-operators, because this method allows to pick out the divergences and then to cancel them in the Serre and “nonstandard” Serre relations. This problem will be considered in the separate paper.

This allows us to represent the $e_{-\alpha_i}$ generators by the $(q - q^{-1})^{-1} \int_{u_1}^{u_2} W_{\alpha_i}$ operators. It was shown in [22] that the corresponding reduced R-matrix \bar{R} , which we will denote $\bar{\mathbf{L}}^{(q)}(u_2, u_1)$ enjoys the P-exponential property, satisfying the following functional relation:

$$\bar{\mathbf{L}}^{(q)}(u_3, u_1) = \bar{\mathbf{L}}^{(q)}(u_3, u_2) \bar{\mathbf{L}}^{(q)}(u_2, u_1). \quad (53)$$

However, in the supersymmetric case when the fermion operators appear the associated singularities in the operator products does not allow us to write it as usual, in terms of ordered integrals. That's why we have called it "quantum" P-exponential, which in our case can be written in the following way:

$$\bar{\mathbf{L}}^{(q)}(u_2, u_1) = \text{Pexp}^{(q)} \int_{u_1}^{u_2} du \left(\sum_{k=0,1,2} W_{\alpha_k}(u) e_{\alpha_k} \right) \quad (54)$$

One can show that $e^{\sum_{i=1,2} \pi i p_i h_{\alpha_i}} \bar{\mathbf{L}}^{(q)}(2\pi, 0) = \mathbf{L}^{(q)}$ satisfy the RTT-relation [4], [5]:

$$\begin{aligned} & \mathbf{R}(\lambda\mu^{-1}) \left(\mathbf{L}^{(q)}(\lambda) \otimes \mathbf{I} \right) \left(\mathbf{I} \otimes \mathbf{L}^{(q)}(\mu) \right) \\ &= \left(\mathbf{I} \otimes \mathbf{L}^{(q)}(\mu) \right) \left(\mathbf{L}^{(q)}(\lambda) \otimes \mathbf{I} \right) \mathbf{R}(\lambda\mu^{-1}), \end{aligned} \quad (55)$$

where the dependence on λ, μ means that we are considering $\mathbf{L}^{(q)}$ -operators in the evaluation representation of $sl_q^{(1)}(2|1)$. We note also, that $\mathbf{M}^{(q)} = e^{\sum_{i=1,2} \pi i p_i h_{\alpha_i}} \bar{\mathbf{L}}^{(q)}$ satisfies the reflection equation [21]:

$$\tilde{\mathbf{R}}_{12}(\lambda\mu^{-1}) \mathbf{M}_1^{(q)}(\lambda) F_{12}^{-1} \mathbf{M}_2^{(q)}(\mu) = \mathbf{M}_2^{(q)}(\mu) F_{12}^{-1} \mathbf{M}_1^{(q)}(\lambda) \mathbf{R}_{12}(\lambda\mu^{-1}), \quad (56)$$

where $F = K^{-1}$, the Cartan's factor from the universal R-matrix, $\tilde{\mathbf{R}}_{12}(\lambda\mu^{-1}) = F_{12}^{-1} \mathbf{R}_{12}(\lambda\mu^{-1}) F_{12}$ and labels 1,2 denote the position of multipliers in the tensor product. This leads to the quantum integrability relation:

$$[\mathbf{t}^{(q)}(\lambda), \mathbf{t}^{(q)}(\mu)] = 0, \quad (57)$$

where $\mathbf{t}^{(q)}(\lambda) = \text{str} \mathbf{M}^{(q)}(\lambda)$.

Now we want to show that in the classical limit when $q \rightarrow 1$ $\mathbf{L}^{(q)}$, $\mathbf{M}^{(q)}$ give the auxiliary \mathbf{L} -matrix and the monodromy matrix correspondingly. Note also, that as usual, quantum universal R-matrix gives the classical r-matrix in the limit $q \rightarrow 1$, therefore the classical limit of $\mathbf{L}^{(q)}$ -operator gives the realization of classical r-matrix by means of classical counterparts of the corresponding vertex operators.

In order to find the classical limit, let's decompose $\bar{\mathbf{L}}^{(q)}$ in the following way [2]:

$$\bar{\mathbf{L}}^{(q)}(2\pi, 0) = \lim_{N \rightarrow \infty} \prod_{m=1}^N \bar{\mathbf{L}}^{(q)}(x_m, x_{m-1}), \quad (58)$$

where we divided the interval $[0, 2\pi]$ into infinitesimal intervals $[x_m, x_{m+1}]$ with $x_{m+1} - x_m = \epsilon = 2\pi/N$. Let's find the terms that can give contribution of the first order in ϵ in $\bar{\mathbf{L}}^{(q)}(x_m, x_{m-1})$. In this analysis we will need the operator products of vertex operators. The ones giving nontrivial terms are:

$$\xi_1(u) \xi_2(u') = \frac{i\beta^2}{(iu - iu')} + \sum_{k=0}^{\infty} c_k(u) (iu - iu')^k,$$

$$\begin{aligned}
& : e^{a\phi_1(u)} :: e^{b\phi_2(u')} := (iu - iu')^{\frac{-ab\beta^2}{2}} (: e^{(a\phi_1(u)+b\phi_2(u))} : + \\
& \sum_{k=1}^{\infty} d_k(u)(iu - iu')^k), \\
& \phi'_1(u) : e^{b\phi_2(u')} := \frac{-ib\beta^2 : e^{b\phi_2(u)} :}{2(iu - iu')} + \sum_{k=0}^{\infty} f_k(u)(iu - iu')^k, \\
& \phi'_2(u) : e^{b\phi_1(u')} := \frac{-ib\beta^2 : e^{b\phi_1(u)} :}{2(iu - iu')} + \sum_{k=0}^{\infty} f_k(u)(iu - iu')^k. \quad (59)
\end{aligned}$$

In the previous cases considered in [2] only two types of terms gave the contribution of the order ϵ in $\bar{\mathbf{L}}^{(q)}(x_{m-1}, x_m)$ when $q \rightarrow 1$. The first type consists of operators of the first order in W_{α_i} and the second type is formed by the operators, quadratic in W_{α_i} , which give contribution of the order $\epsilon^{1 \pm \beta^2}$ by virtue of operator product expansion. This second type contributions correspond to the composite roots which are equal to the sum of two simple roots.

In the present case we will show that there are contributions of the composite roots which are equal to the sum of three and even four simple roots providing the desired terms in the classical expression (42-43). First, let's consider the quadratic terms corresponding to the negative roots $-\alpha_1 - \alpha_2, -\delta + \alpha_2, -\delta + \alpha_1$. The commutation relations between vertex operators on a circle are:

$$\begin{aligned}
W_{\alpha_i}(u)W_{\alpha_j}(u') &= -q^{-1}W_{\alpha_j}(u')W_{\alpha_i}(u) \quad u > u' \quad (i, j = 1, 2 \quad i \neq j) \\
W_{\alpha_i}(u)W_{\alpha_0}(u') &= qW_{\alpha_0}(u')W_{\alpha_i}(u) \quad u > u' \quad (i = 1, 2) \\
W_{\alpha_0}(u)W_{\alpha_i}(u') &= qW_{\alpha_i}(u')W_{\alpha_0}(u) \quad u > u' \quad (i = 1, 2)
\end{aligned} \quad (60)$$

This allows us to write the generators corresponding to the negative composite roots $-\delta + \alpha_2, -\delta + \alpha_1$ in the following way:

$$\begin{aligned}
[e_{-\alpha_0}, e_{-\alpha_1}]_q &= \frac{1}{q - q^{-1}} \int_{x_{m-1}}^{x_m} du W_{\alpha_1}(u) \int_{x_{m-1}}^u du' W_{\alpha_0}(u') \\
[e_{-\alpha_2}, e_{-\alpha_0}]_q &= \frac{1}{q - q^{-1}} \int_{x_{m-1}}^{x_m} du W_{\alpha_0}(u) \int_{x_{m-1}}^u du' W_{\alpha_2}(u') \quad (61)
\end{aligned}$$

The exponents of the corresponding q-exponentials from (48) are equal to

$$\begin{aligned}
& \int_{x_{m-1}}^{x_m} du W_{\alpha_1}(u) \int_{x_{m-1}}^u du' W_{\alpha_0}(u') [e_{\alpha_1}, e_{\alpha_0}]_{q^{-1}} \\
& \int_{x_{m-1}}^{x_m} du W_{\alpha_0}(u) \int_{x_{m-1}}^u du' W_{\alpha_2}(u') [e_{\alpha_0}, e_{\alpha_2}]_{q^{-1}} \quad (62)
\end{aligned}$$

In the classical limit ($\beta^2 \rightarrow 0$) their contribution due to the operator products (this calculation is similar to the calculation given in [13]) is:

$$\int_{x_{m-1}}^{x_m} du \left(-\frac{i}{\sqrt{2}} \xi_2 e^{\phi_2} [e_{\alpha_2}, e_{\alpha_0}] - \frac{i}{\sqrt{2}} \xi_1 e^{\phi_1} [e_{\alpha_0}, e_{\alpha_1}] \right) \quad (63)$$

The contribution of the root $-\alpha_1 - \alpha_2$ is equal to

$$\int_{x_{m-1}}^{x_m} du (-e^{-\phi_1 - \phi_2} [e_{\alpha_1}, e_{\alpha_2}]) \quad (64)$$

In this case the commutator of the integrated vertex operators cannot be rewritten in terms of the ordered integrals as it was above, however using the fact that the integrated vertex operators should be radially ordered (that is e.g. the product $e_{-\alpha_i} e_{-\alpha_j}$ should be written as $\int_{x_{m-1}}^{x_m} du W_{\alpha_i}(u - i0) \int_{x_{m-1}}^{x_m} du' W_{\alpha_j}(u' + i0)$) and the well known relation $\frac{1}{x+i0} - \frac{1}{x-i0} = -2i\pi\delta(x)$, we obtain this formula repeating calculation of [2]. Thus on this “quadratic” level we are in agreement with the classical expression.

Now let's look at the contribution corresponding to the composite roots $e_{\delta}^{(i)}$ and $2\delta - \alpha_1 - \alpha_2$. First we consider the pure imaginary roots: $e_{-\delta}^{(1)} = [[e_{-\alpha_2}, e_{-\alpha_0}]_q, e_{-\alpha_1}]$, $e_{-\delta}^{(2)} = [[e_{-\alpha_0}, e_{-\alpha_1}]_q, e_{-\alpha_2}]$. To calculate their contribution in the classical limit we at first need to take the terms which give contribution of the order $\epsilon^{1+\beta^2}$ from $[e_{\alpha_2}, e_{\alpha_0}]_q$ and $[e_{\alpha_0}, e_{\alpha_1}]_q$, which are proportional to $\int dud\theta e^{\Phi_2}$ and $\int dud\theta e^{\Phi_1}$. Then we consider their supercommutators with correspondingly e_{α_1} and e_{α_2} . Rewriting these supercommutators in terms of ordered integrals as it was above we find, following the calculations in [13] that the contribution of the associated exponent of q-exponential is:

$$\int_{x_{m-1}}^{x_m} du (-[e_{\alpha_2}, [e_{\alpha_0}, e_{\alpha_1}]] - [e_{\alpha_1}, [e_{\alpha_2}, e_{\alpha_0}]]) \quad (65)$$

Similarly, $e_{2\delta - \alpha_1 - \alpha_2}$ generator is expressed as the q-commutator of $[e_{\alpha_2}, e_{\alpha_0}]_q$ and $[e_{\alpha_0}, e_{\alpha_1}]_q$, thus, leaving again their parts of order $\epsilon^{1+\beta^2}$ and using the formula $\frac{1}{x+i0} - \frac{1}{x-i0} = -2i\pi\delta(x)$ as it was explained above we obtain that the classical contribution of the first order in ϵ is:

$$\int_{x_{m-1}}^{x_m} du (-[[e_{\alpha_0}, e_{\alpha_1}], [e_{\alpha_2}, e_{\alpha_0}]] e^{\phi_1 + \phi_2}) \quad (66)$$

Therefore, gathering all the terms we obtain that

$$\begin{aligned} \lim_{q \rightarrow 1} \bar{\mathbf{L}}^{(q)}(x_m, x_{m-1}) &= 1 + \int_{x_{m-1}}^{x_m} du \left(\sum_{k=0,1,2} W_{\alpha_k}(u) e_{\alpha_k} - \frac{i}{\sqrt{2}} \xi_2 [e_{\alpha_2}, e_{\alpha_0}] e^{\phi_2} \right. \\ &\quad \left. - \frac{i}{\sqrt{2}} \xi_1 [e_{\alpha_0}, e_{\alpha_1}] e^{\phi_1} - [e_{\alpha_1}, e_{\alpha_2}] e^{-\phi_1 - \phi_2} - [[e_{\alpha_0}, e_{\alpha_1}], [e_{\alpha_2}, e_{\alpha_0}]] e^{\phi_1 + \phi_2} - \right. \\ &\quad \left. [e_{\alpha_2}, [e_{\alpha_0}, e_{\alpha_1}]] - [e_{\alpha_1}, [e_{\alpha_2}, e_{\alpha_0}]] \right) + O(\epsilon^2) \end{aligned} \quad (67)$$

Collecting all the infinitesimal $\bar{\mathbf{L}}$ operators and multiplying on the appropriate Cartan factors we obtain the desired classical expressions for the \mathbf{L} -operators and the monodromy matrix.

8 Conclusion

The construction of the monodromy matrix given in this paper follows the way introduced in [9], [15], [1], [2]. It is shown that the quantum version of auxiliary \mathbf{L} -matrix coincides with the universal \mathbf{R} -matrix with the lower Borel algebra represented by the appropriate vertex operators. This construction allows also to demonstrate that the supersymmetry generators commute with the supertrace of the monodromy matrix, i.e. they can be included in the series of integrals of motion. Really, the commutators of the supersymmetry generator

$$G_0 = \beta^{-2} \sqrt{2} i^{-1/2} \int_0^{2\pi} du \phi'(u) \xi(u)$$

in the case of $N=1$ SUSY KdV and two supersymmetry generators

$$G_0^+ = \beta^{-2} \sqrt{2} i^{-1/2} \int_0^{2\pi} du \phi'_1(u) \xi_2(u), \quad G_0^- = \beta^{-2} \sqrt{2} i^{-1/2} \int_0^{2\pi} du \phi'_2(u) \xi_1(u)$$

in the case of $N=2$ SUSY KdV, with vertex operators from the corresponding quantum P -exponent reduce to the total derivatives and therefore (according to the results of [2],[15]) they commute with the transfer matrices. The same way one can show that the transfer matrices commute with the zero mode of the $U(1)$ current which is present in the $N = 2$ superconformal algebra (the quantum version of the V field from (35)).

This result has very important consequences. The SUSY $N=1$ KdV model can be used to describe the integrable structure of supersymmetric perturbed nonconformal two-dimensional integrable models with perturbation described by the vertex operator $\int d\theta e^{-\Phi}$ with the dimension $h_{1,3}$ from the Kac table.

In the case of $N=2$ SUSY KdV the situation is more interesting. If we make the twist transformation [23] in the underlying $N=2$ superconformal algebra we obtain one of the generators G_0^\pm to become a BRST operator. It means that the transfer-matrices become BRST-exact providing the infinite series of pairwise commuting “physical” integrals of motion (of the zero ghost number provided by the zero mode of $U(1)$ current). This gives possibility to study the two-dimensional topological models and their integrable perturbations by means of methods of integrable field theories e.g. the most popular one, the so-called Quantum Inverse Scattering Method [4], [5].

Acknowledgments

This work was supported by CRDF Grant RUM1-2622-ST-04 and RFBR Grant 05-01-00922. A.M.Z. is grateful to the Dynasty foundation for support and to N. Reshetikhin, M. Semenov-Tian-Shansky and A. Tseytlin for encouragement.

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